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Generalization and Cogitation of Leibniz Derivative Rule

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ABSTRACT

This study is about the generalization of Leibniz's derivative rule, which has been done as research. That is, obtaining an extension of the derivative of the nth order of the product of the nth function, which is the successive derivatives up to the nth order. Leibniz's rule is the derivative of the nth order of the product of two functions, which is in the form of an expansion and has successive derivatives up to the nth order. First, the generalization of a theorem in mathematics is explained. Also, the derivative of the product of two and more functions, then the derivatives of the first to the nth order of a function and the rule of Leibniz's derivative are discussed and we have an overview of the generalization of this rule. The results show that the relationship between the order of rivatives of functions and coefficients in the general sentence of the generalized rule is the same as the relationship of owers and coefficients in the general sentence of the expansion of polynomials. In order to obtain the derivative of higher order in the multiplication of several functions, less process is easily used.

Keywords: Derivative of the product of two or more functions, Leibniz rule, and Generalization of Leibniz rule.

INTRODUCTION:

The concept of the derivative in its current form was first developed by Newton in 1666, and a few years after him, by Gottfried Wilhelm Leibniz independently of each other. Leibniz, the greatest comprehensive genius of the 17th century, was Newton's rival in the inventing calculus. The general opinion today is that each of them discovered calculus independently of the other. Although Newton's discovery was made earlier, Leib-niz published the results earlier. He has extracted many of the rules of derivation, which a student learns in the beginning of an introductory course in calculus.

The derivative of function f can also be represented by f'. This symbol emphasizes that f' is a new function obtained by deriving from function f and its value at $\boldsymbol{x}$ is given by $f'(x)$. This symbol was used by Joseph Louis Lagrange in 1770 AD. Derivatives of higher orders are shown as the f'(first derivative), f'' (second derivative), and f''' (third derivative), $f^{(4)}$ (fourth derivative) and .... $f^{(n)}$ (nth derivative). The rule for finding the derivative of the product of two functions is still called Leibniz's rule.

A generalization of a mathematical theorem is a theorem that gives a wider result with the same premise of the theorem. So that the generalized theorem is obtainned from it. Through the generalization of a theorem, it is possible to look at that theorem from a broader angle. Most mathematical works, even the results of outstanding mathematicians, are usually generalizations of existing works and concepts. There are two types of the patterns for generalization: "result-based generalization" and the "process-based generalization". Generalization based on the result is a generalization that is obtained by recognizing a general pattern of the result itself. Process-based generalization is a generalization resulting from a process that ends in a result, that is, a way that contains a specific chain of steps dependent on previous results (Niazai *et al*., 2022).

Leibniz's rule and its generalization are used to obtain the derivative of higher order in the product of two or more functions. For example, in functions such as$u = sinx$, $v = lnx$, and $w = x^2$, if we want to obtain the derivative of higher orders in their product ($y = uvw$), we use the generalization of Leibniz's rule for the product of three functions. This method makes us go through a less process in obtaining the derivative of higher orders, especially when the product of several functions cannot be written as a sum of several terms.

The derivative of the product of two or more functions and derivatives of the first to nth order:
Suppose u and v are two functions of x and we have: $y = u.v$ then the derivative of the function y will be: (first order derivative): $y' = (uv)'$

Leibniz's symbolization method:

$$\frac{d(u.v)}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

We assume that u and v are continuous. The derivative of their product will be: $y' = u'v + uv'$
We also have several functions:

$$y = uvw$$
$$y' = (uvw)' \to y' = u'vw + uv'w + uvw'$$

If the function is continuous and differentiable, we have: $y' = f'(x)$, if $y' = f'(x)$ is also differentiable, the second derivative of the original function $y = f(x)$ can be found:

$$y'' = f''(x) = (f'(x))' \qquad y'' = (y')'$$

Similarly, if $y'' = f''(x)$ is differentiable, the third derivative of the function $f(x)$ is as follows:

$$y''' = (f''(x))' = (y'')'$$

Derivatives of higher order can be defined as follows:

$$y^{(n)} = (y^{(n-1)})' = \frac{d^n y}{dx^n}$$

Leibniz rule (derivative):
If $y = u.v$, where u and v are functions of x. we can say that the nth order derivative of y with respect to x is obtained from the following equation:

$$(I)\ y^{(n)} = uv^{(n)} + nu'v^{n(n-1)} + \frac{n(n-1)}{2!}u''v^{(n-2)} + \frac{n(n-1)(n-2)}{3!}u'''v^{(n-3)} + \cdots + u^{(n)}v$$

The relationship (I) is also written as follows:

$$y^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

The sum of derivative orders of functions u and v in each term is equal to (n):

$$(k) + (n - k) = n$$

The coefficient of each term in relation (I) is equal to: $\frac{n!}{k!(n-k)!}$

The general expression of the relation (I) is as follows:
$Tp = \frac{n!}{k!(n-k)!}u^{(k)}v^{(n-k)}$

Generalization of Liebnitz's rule for n function:
Suppose u_i is a function of x, we have $u_i = f_i(x)$ and
$y = \prod_{i=1}^{n} u_i \qquad i \in N$
$y = u_1u_2u_3 \dots u_n$
If the functions $u_1u_2u_3 \dots u_n$ are continuous and differentiable, we have:

$$y' = (u_1u_2u_3 \dots u_n)'$$
$$y' = u_1'u_2u_3 \dots u_n + u_1u_2'u_3 \dots u_n + u_1u_2u_3' \dots u_n + \cdots + u_1u_2u_3 \dots u_n'$$
$$\begin{aligned} y'' = {} & (2u_1'u_2'u_3 \dots u_{n-1} + 2u_1'u_3'u_2 \dots u_n + 2u_1'u_4'u_2u_3 \dots u_n + \cdots + 2u_1'u_n'u_2u_3 \dots u_{n-1}) \\ & + (u_1''u_2u_3 \dots u_n + u_1u_2''u_3 \dots u_n + u_1u_2u_3'' \dots u_n + \cdots + u_1u_2u_3 \dots u_n'') \\ & + \left(2u_1u_2'u_3' \dots u_n + 2u_1u_2'u_4'u_3 \dots u_n + 2u_1u_2'u_5'u_4u_3 \dots u_n + \cdots + 2u_1u_2'u_n'u_3u_{(n-1)}\right) \\ & + (2u_1u_2u_3'u_4'u_5u_6 \dots u_n + 2u_1u_2u_3'u_5'u_4u_6 \dots u_n + \cdots + 2u_n'u_3'u_1u_2 \dots u_{n-1}) + \cdots (2u_n'u_1'u_2 \dots u_{n-1} \\ & + 2u_n'u_2'u_1u_3 \dots u_{n-1} + 2u_n'u_3'u_1u_2 \dots u_{n-1} + \cdots + 2u_n'u_{n-1}'u_1u_2u_3 \dots u_{n-2}) \end{aligned}$$

We have for the derivative of the nth order of function $y = \prod_{i=1}^{n} u_i$: ($i \in N$)

$$(II)\ y^{(n)} = \left(u_1u_2u_3 \dots u_n^{(n)} + u_1u_2 \dots u_{n-1}^{(n)}u_n + \cdots + u_1u_2^{(n)}u_3 \dots u_n + u_1^{(n)}u_2u_3 \dots u_n\right) + u_1^{(n)}u_2u_3 \dots u_n + n\left(u_1u_2 \dots u_{n-1}^{'}u_n^{(n-1)}\right) + n\left(u_1u_2 \dots u_{n-2}^{'}u_{n-1}u_n^{(n-1)}\right) + \cdots + n\left(u_1^{'}u_2 \dots u_{n-1}u_n^{(n-1)}\right) + n(n-1)u_1^{'}u_2^{'}u_3 \dots u_n^{(n-2)} + n(n-1)u_1^{'}u_3^{'}u_2 \dots u_n^{(n-2)} + \cdots + n(n-1)u_1^{'}u_{n-1}^{'}u_2u_3 \dots u_n^{(n-2)} + \cdots$$

From the expansion sentences y'' and the relation (*II*), the general sentence can be concluded:

$$Tp = \frac{n!}{a_1!\,a_2!\,a_3! \dots a_n!} \times u_1^{(a_1)}u_2^{(a_2)}u_3^{(a_3)} \dots u_n^{(a_n)}$$

$j \in N, a_j \in Z \qquad a_1 + a_2 + a_3 + \cdots + a_n = n$

$a_j \geq 0$

a_j is the derivative of functions u_i:

Order of the derivative of the function $a_1 : u_1$

Order of the derivative of the function $a_2 : u_2$

Order of the derivative of the function $a_3 : u_3$

....

....

Order of the derivative of the function $a_n : u_n$

-The zeroth order of the derivative of a function is that function itself.

$$\begin{cases} u_i^{(0)} = u_i \\ a_j = 0 \end{cases}$$

$$(III)\ Tp = \frac{n!}{\prod_{j=1}^{n} aj!} \times \prod_{\substack{i=1 \\ j=1}}^{n} u_i^{(aj)}$$

The general sentence of relation (*II*)

- Note (1): The general sentence of relation (*II*) is concluded through process-based generalization.

Examples:

Leibniz's rule can be concluded from relation (*II*) (from its general sentence formula):

If we omit $u_3u_4 \dots u_n$ functions in the general sentence of relation (*II*), we will have:

$$Tp = \frac{n!}{a_1!\,a_2!} \times u_1^{(a_1)}u_2^{(a_2)}$$

$$Tp = \frac{n!}{k!\,(n-k)!}u_1^{(k)}u_2^{(n-k)} \qquad k \geq 0$$

$$\begin{cases} a_1 + a_2 = n \\ a_1 = k \end{cases} \rightarrow a_2 = n - k$$

$$\rightarrow y^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u_1^{(k)}u_2^{(n-k)}$$

With a little precision, we can see that the relationship between the coefficients and the order of the derivative of the functions in the expansion of Leibniz's rule is the same as the relationship between the coefficients and powers in the expansion of Newton's binomial. Also, the relationship between the derivative order of functions and coefficients in the generalization of Leibniz's derivative rule in the general sentence is the same as the relationship between the coefficient and power in the expansion of polynomials (general sentence):

$$(x_1x_2x_3 + \cdots + x_n)^n$$

$$Tp = \frac{n!}{a_1!a_2!a_3!\dots a_n!}x_1^{a_1}x_2^{a_2}x_3^{a_3} \dots x_n^{a_n}\ (IV)$$

$$a_1 + a_2 + a_3 + \cdots + a_n = n$$

Example: if the functions u, v and w are functions of the variable x and we have $y = uvw$, the expan-sion of the third derivative of the function y is equal to

$$y^{(3)} = (uvw)'''$$

$$y = u_1u_2 \rightarrow y^{(n)} = (u_1u_2)^{(n)}$$

$$y^{(3)} = u'''vw + uv'''w + uvw''' + 6u'v'w' + 3u''v'w + 3u'v''w + 3uv'w'' + 3u'vw'' + 3u''vw' + 3v''uw'$$

Note (2): It may be possible to conclude the general sentence of the relation (*II*) through the generalization based on the result. The relation (*I*) is the same as Newton's binomial expansion. The general sentence of relation (*II*) is also the same as the general sentence of polynomial expansion, that is, relation (*IV*). Of course, we are more interested in process-based generalization here.

CONCLUSION:

This research aimed to obtain a generalization of Leibniz's derivative rule for finding higher-order derivatives of products of multiple functions. Through mathematical analysis, a generalized formula was derived that allows efficiently computing the nth derivative of a product of n functions. The key finding is that the coefficients in the generalized formula correlate to the derivative orders of each function in the product, analogous to how coefficients correlate with powers in polynomial expansions. Specifically, the coefficient of each term is n!/(a1!a2!...an!) where a1 to an denote the orders of derivatives of functions u1 to un. Additionally, Σaj = n, meaning the total order of derivatives in each term sums to n. This generalized rule was shown to reduce to Leibniz's formula for two functions. Further, an example demonstrated its application for a product of three functions. Compared to differentiating such products term-by-term, the new formula requires significantly less algebraic manipulations.

In conclusion, this research presented a broader, systematic rule for obtaining higher-order derivatives of products of multiple differentiable functions. It has theoretical and practical implications for fields relying on differentiation like physics, engineering, and the optimization. Further studies can explore if additional generalizations are possible using alternative approaches. With analytical proofs and illustrative examples provided, this work contributes a useful extension to foundational calculus rules.

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CONFLICTS OF INTEREST:

The author have declared no conflict of interest.

REFERENCES:

1) Leibniz, G. W. (2012). The early mathematical manuscripts of Leibniz. *Courier Corporation*. https://books.google.com.my/books/about/The_Early_Mathematical_Manuscripts_of_Le.html?id=tCmp_c3Q9S8C&redir_esc=y
2) Maral Ruin, Rostam Mohammadian, Noorullah Nejad Sadeghi, (2014). Math Culture and Thought, No. **56**, Retrieved from: *mct.iranjournals.ir*
3) Mohammad Hasan Bijanzadeh, Vahid Alamian, Gholamali Farshadi, (2013). calculus. *Iran, Publishing Company*. ISBN 978-964-05-2009
4) Niazai S, Rahimzai AA, Danesh M, and Safi B. (2022). Numerical solution of diffusion equation with caputo time fractional derivatives using finite-difference method with Neumann & Robin boundary conditions, *Aust. J. Eng. Innov. Technol.*, **4**(5), 95-108. https://doi.org/10.34104/ijmms.022.0950108
5) Niven, I. (1965). Mathematics of Choice: Or, How to Count Without Counting, **15**, MAA. https://books.google.com.my/books/about/Mathematics_of_Choice.html?id=IOO0s7-ScZEC&redir_esc=y
6) W. Eves, H. (1983). An Introduction to the History of Mathematics, Fifth Edition, Saunders College *Publishing*, pp. 108-113. https://books.google.com.my/books/about/An_Introduction_to_the_History_of_Mathem.html?id=PXvwAAAAMAAJ&redir_esc=y

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